MATH4210: Financial Mathematics Tutorial 11

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r is an annullized rate

Let r be the interest rate. Suppose that you place $\$x_0$ in an account in a bank. After n years, you will have the amount

- $y_n = x_0(1 + nr)$ if the interest rate is the simple interest rate.
- $y_n = x_0(1 + r)^n$ if the interest rate is the annual compound interest rate.

  $\$ x_0$ $\quad$ $x_0 + (r \cdot x_0)$ $\quad$ $(x_0 + r x_0) r + x_0$

  1 year

- $y_n = x_0(1 + \frac{\boxed{r}}{m})^{mn}$ if the interest rate is the compound interest rate and compound m times per annul.

  x_0 $\quad$ $x_0 + x_0 \cdot \frac{r}{2}$ $\quad$ $(x_0 + x_0 \cdot \frac{r}{2}) \frac{r}{2} + x_0$

  1/2 year. $\quad$ 1 year

- $y_n = x_0 e^{nr}$ if the interest rate is the continuous compound interest rate.

$y_n = x_0 + \text{interest income}$

Present Value (Time Value of Money)

We can always use $\$x_0$ now as principal in a ~~risk-free~~ investment at (continuous compound interest) rate $r > 0$. It guarantees the amount

$$\boxed{x_0 e^{rt}} > x_0$$

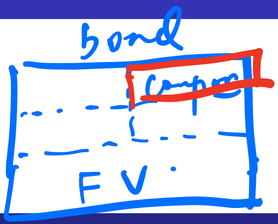
at time t. Equivalently, if we deposit $\boxed{\$xe^{-rt}}$ at the bank, we get $\$x$ at time t, thus

$$\text{We call } \boxed{xe^{-rt}} \text{ the present value (PV) of } x,$$

which is also called the discounted value of x at the future time t, and the factor e^{-rt} is called the discount factor.

PV of x is $x \cdot$ Discount $\left(e^{-rt}, \frac{1}{(1+rt)}, \frac{1}{(1+r)^t}\right)$

Interest Rate



· PV, Face Value, coupon — 4 year coupon - bond

price (par/principal)

Proposition

Present value of a financial derivative is the sum of all discounted future cash flows:

$$PV = \sum_i \frac{CashFlow_i}{Dicount_i} \overset{eg}{=} \sum_{i=1}^{n} \left(\frac{coupon_i}{e^{ri}}\right) + \frac{Par}{e^{n \cdot r}}$$

(coupon - bond)

Question

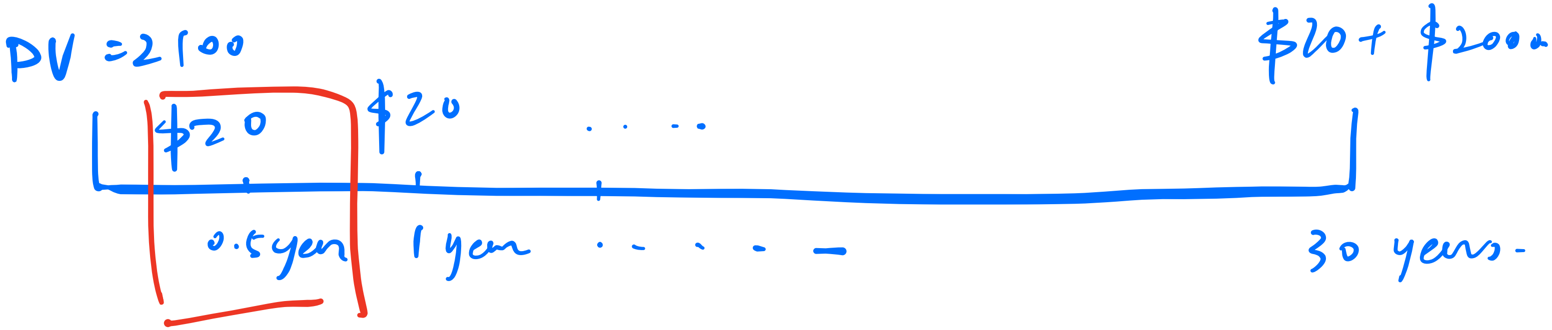
a) Find the value of a 10-year zero-coupon bond of face value \$100 *if the the annual simple interest rate is* 2%.

b) Find the face value of a 10-year zero-coupon bond if it is issued for \$100 *and the continuous compound interest rate is* 3%.

a): $PV = \frac{\text{Cash flow}}{\text{Discount}} = \frac{100}{(1+2\% \cdot 10)}$

b) $PV = \$100$. $r = 3\%$, no coupon.

$PV = \$100 = \frac{FV}{e^{3\% \boxed{10}}} \Rightarrow FV = \$100 \times e^{30\%}$.



Question

T FV

Consider a 30-year \$2000 bond, that has coupons every 1/2 year in the amount of \$20, for a total of 60 times until 30 years at which time you receive \$2020. The bond price is \$2100. What is the yield (i.e. internal rate of return) if the rate is continuously compounding?

30×2

20 + par

r?

$$2100 = PV = \frac{20}{e^{0.5r}} + \frac{20}{e^{1.0r}} + \frac{20}{e^{1.5r}} + \cdots + \frac{20+2000}{e^{30r}}$$

$$= \sum_{i=1}^{30} \frac{20}{e^{0.5\cdot i\cdot r}} + \frac{2000}{e^{30r}}$$

set $x = e^{0.5r}$

$$2100 = \sum_{i=1}^{30} \boxed{\frac{20}{x^{i}}} + \frac{2000}{x^{60}} \Rightarrow a\cdot x^{60} + b\cdot x^{1} + c = 0$$

$x = 1,\ x = 0.?$

Present Value

Question

Pricing a coupon bond: consider a 2-year \$2000 bond, that has coupons every 1/2 year in the amount of \$50, for a total of four times until 2 years at which time you receive \$2050. Suppose the continuous compound interest rate is r. What is its price of the bond?

$$PV = \sum_{i=1}^{4} \frac{50}{e^{0.5\cdot r\cdot i}} + \frac{2000}{e^{2\cdot r}}$$

Question

An imaginary nice government that does not exists on this planet promises to pay you (and your descendant) $10,000 immediately and the same amount every year perpetually. If the the compound annual interest rate is 2.5%, what is its present value of this plan?

$$PV = \sum \frac{\text{Cash flow}}{\text{Discount}} = \sum_{i=0}^{+\infty} \frac{\$10000}{(1+2.5\%)^i} = 10000 \sum_{i=0}^{+\infty} \frac{1}{(1.025)^i}$$

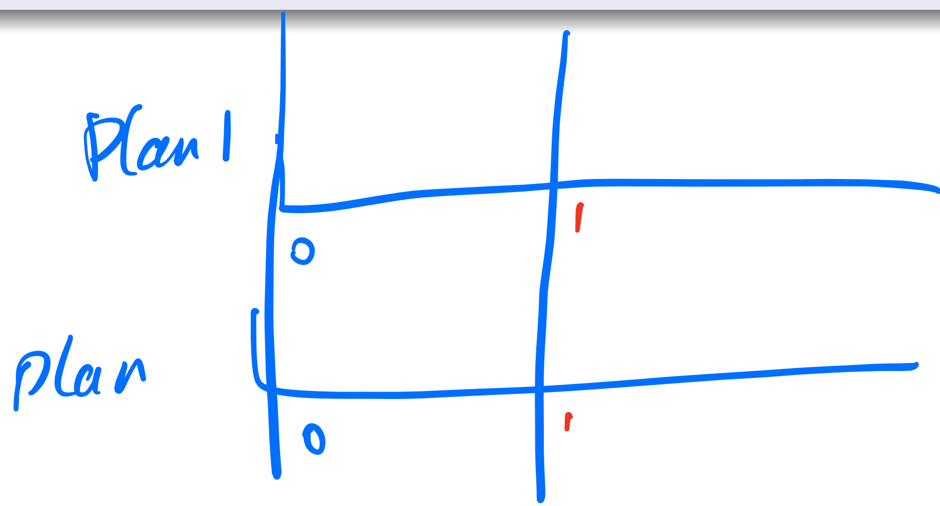
$$= 10000 \cdot \left(\frac{1}{1-\frac{1}{1.025}} \right)$$

Question

Joyce wants to use a land to build a church. The government requires her to pay the nominal rent 1,000 every year perpetually. A bank offers a plan: Joyce pay the bank 50,000 at once and the bank promises to pay 1,000 to the government every year. Suppose the discrete annual compound interest is 2%. Should Joyce accept this offer? (Unit: \$)

$$PV_1 = 1000\left(\frac{1}{1-\frac{1}{1.02}}\right)$$

$$PV_2 = 50000$$



Question (Example on Slides 5)

(a).Suppose that we have three European call options with the same maturity T in the financial market whose price at time $t = 0$ are:

$C_2(T = T_0)$

$\vee\!\!|$

$C_2(T = 2T_0)$

$P_2(T = T_0)$

$\wedge\!\!|$

$P_2(T = 2T_0)$

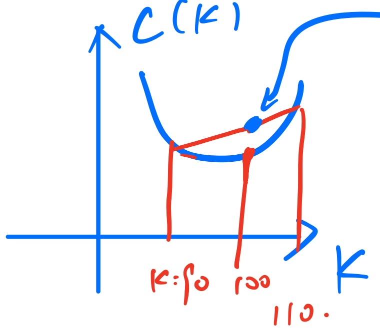
$$C_1(K = 90) = 10$$
$$C_2(K = 100) = 9$$
$$C_3(K = 110) = 7.$$

What if C_1, C_3 unchanged.

$C_2(K = 100, T' = 2T)$
$= \$9.$

Suppose the interest rate is zero. Construct the arbitrage strategy.
(b). At $t = 0$, the underlying asset $S_0 = 100$. We keep C_1 and C_3 the same. But We don't have C_2, instead there is a European put option with the same setting such that $P_2(K = 100) = 9$. Find the arbitrage strategy.

$C(K)$ is the function of strike price is convex.



$$\frac{C(K=90) + C(K=110)}{2} \geq C(K=100)$$

$$\underbrace{\frac{C_1 + C_3}{2}}_{\text{under priced}} = \frac{10+7}{2} = 8.5 < \underbrace{C_2 = 9}_{\text{overpriced.}}$$

So either $\frac{C_1+C_3}{2}$ is lower than actual value

or. C_2 is higher than actual value

So we buy low and sell high:

$$\pi = \frac{C_1+C_3}{2} - C_2$$

or, $\pi = C_1 + C_3 - 2C_2$.

(b): Use the put-call parity to construct C_2 using P_2.

$$C_2 = P_2 + S_0 - K = P_2 \cancel{+100} \cancel{-100}$$

$$= P_2$$